Subgraph lemma

Fix constants $\epsilon, \delta > 0$ and let G be a graph with partition $V_1, \dots, V_r$ from the regularity lemma. In many applications our first step is to "clean up" G by removing all edges that are either inside the partition classes, between pairs of classes that are not ϵ-regular, or between classes that have density $< \delta$. This procedure results in a graph G' (depending on ϵ and δ).

1: Find a function f such that $e(G) \le e(G') + f(\frac{1}{r}, \epsilon, \delta)n^2$.

Solution: We calculate what edges are deleted. Inside each part, we get $r \cdot \binom{\lceil n/r \rceil}{2} \le \frac{1}{r}n^2$. Between parts, that are not ϵ-regular, we get $\varepsilon r^2 \left\lceil \frac{n}{r} \right\rceil^2 \le 2\epsilon n^2$. Finally, if density is less than δ, we get $\delta \binom{r}{2} \left\lceil \frac{n}{r} \right\rceil^2 \le 2\delta n^2$.

$$e(G) \le e(G') + \left(\frac{1}{r} + 2\epsilon + \delta\right) n^2.$$

Now define the (ϵ, δ)-**reduced graph**[1] R of G as follows. The vertices of R are the partition classes $V_1, \dots, V_r$ and two classes V_i, V_j are connected by an edge if the pair is ϵ-regular and has density $\ge \delta$. That is, R has an edge if the two corresponding classes have edges in G'. This implies that

$$e(R)\delta \left\lfloor \frac{n}{r} \right\rfloor^2 \le e(G') \le e(R) \left\lceil \frac{n}{r} \right\rceil^2.$$

Many properties of the original graph G are inherited by the reduced graph R.

The embedding lemma has the following consequence for the reduced graph.

Lemma 1 (Subgraph lemma[2])**.** *Suppose F is an f-vertex $(k+1)$-chromatic graph. If G is F-free and $\delta \ge 2\epsilon^{1/f}$, then the (ϵ, δ)-reduced graph R is K_{k+1}-free.*

Our goal is to use the regularity lemma to prove a version of stability theorem for extremal graphs.. We will use the following stability theorem for K_{k+1}.

Theorem 2 (Füredi, 2010)**.** *Suppose G is an n-vertex K_{k+1}-free graph with $e(G) = e(T_k(n)) - t$. Then G can be obtained from a complete k-partite graph (on n vertices) by adding and removing at most $3t$ total edges.*

We are now ready to prove a version of first stability theorem.

Theorem 3 ("weak" first stability theorem)**.** *For any $\alpha > 0$ and any $(k+1)$-chromatic graph F, there exists $\beta > 0$ and n_0 such that if G is an F-free n-vertex graph with $n > n_0$ and*

$$e(G) > \left(1 - \frac{1}{k}\right)\frac{n^2}{2} - \beta n^2,$$

then G can be obtained from a complete k-partite graph by adding and removing at most αn^2 total edges.

Proof. (Füredi, 2010) Let us apply the regularity lemma to G with parameters ϵ and $m > \frac{1}{\epsilon}$. Then there exists $M = M(\epsilon, m)$ such that any graph on $n \ge M$ many vertices has an equipartition $V_1, \dots, V_r$ with all but at most ϵr^2 pairs of classes are ϵ-regular and $m \le r < M$.

Fix $\delta > 2\epsilon^{1/|F|}$.

[1] Also often called the cluster graph or skeleton (graph).

[2] The condition on the size of δ can be replaced with the much weaker $\delta > \epsilon^{1/k^2}$.

2: Clean up G by removing edges inside each class, not regular pairs or between pairs with density $< \delta$. How much is removed? (Find final answer not containing r)

Solution: Construct a graph G' by removing edges in each class V_i and removing edges between any two classes that are either not ϵ-regular or have density $< \delta$. In total the number of edges removed from G is at most

$$\left(\frac{1}{r} + 2\epsilon + \delta\right) n^2 \leq (3\epsilon + \delta)n^2.$$

Let R be the (ϵ, δ)-reduced graph of G. Each edge in R represents at most $\lceil \frac{n}{r} \rceil^2$ edges in G'.

3: Find a lower bound on $e(R) \left\lceil \frac{n}{r} \right\rceil^2$.

Solution:

$$e(R) \left\lceil \frac{n}{r} \right\rceil^2 \geq e(G') \geq e(G) - (3\epsilon + \delta)n^2 > \left(1 - \frac{1}{k}\right) \frac{n^2}{2} - (3\epsilon + \delta + \beta)n^2.$$

For n large enough we have

$$e(R) \left\lceil \frac{n}{r} \right\rceil^2 \leq e(R) \left(\frac{n}{r}\right)^2 + \epsilon n^2.$$

4: Combining the above bounds and solve for $e(R)$. Use $e(T_k(r))$ in the final answer.

Solution:

$$e(R) > \left(1 - \frac{1}{k}\right) \frac{r^2}{2} - (3\epsilon + \delta + \beta)r^2 \geq e(T_k(r)) - (4\epsilon + \delta + \beta)r^2.$$

Put $t = e(T_k(r)) - e(R) \leq (4\epsilon + \delta + \beta)r^2$ and apply Theorem 2 to R (as it is K_{k+1}-free).

5: What does the application of Theorem 2 to R mean for G'?

Solution: That is, R can be made into a complete k-partite graph by adding or removing at most $3t$ edges. For n large enough this corresponds to changing at most

$$3t \left\lceil \frac{n}{r} \right\rceil^2 \leq 3t \left(\frac{n}{r}\right)^2 + \epsilon n^2 \leq (13\epsilon + 3\delta + 3\beta)n^2$$

edges to make G' into a complete k-partite graph.

6: Finish the proof by considering what edges were removed from G to get G'.

Solution: We removed at most $(3\epsilon + \delta)n^2$ edges to get G' from G so in total we need to add or remove

$$\alpha n^2 = (16\epsilon + 4\delta + 3\beta)n^2$$

edges to make G into a complete k-partite graph. Thus for any given $\alpha > 0$ we can choose β, ϵ, δ each small enough to satisfy the theorem.

$\square$

Lemma 4. *Let H be a k-partite graph with classes $C_1, \dots, C_k$ each of size at most ℓ and let q be the number pairs of distinct classes C_i, C_j such that there is at least one edge between C_i, C_j. If for $2 \le s \le t$, there is no $K_{s,t}$ with the class of size s contained in any C_i, then*

$$e(H) \le \frac{1}{2}(2q)^{1-1/s}\ell^{2-1/s}(t-1)^{1/s}k^{1/s} + q\ell s.$$

Proof. For a vertex $x \in H$, let $d_i(x)$ be the number of neighbors of x in C_i. Furthermore, let P be the set of pairs (x, i) such that x is a vertex in H and $d_i(x) > 0$.

7: Find an upper bound on $|P|$ in terms of q and ℓ and simplify $\sum_{(x,i)\in P} d_i(x)$.

Solution: It is easy to see

$$|P| \le 2q\ell$$

and

$$\sum_{(x,i)\in P} d_i(x) = 2e(G).$$

Now, as in the proof of KST, let us double count the pairs (x, S) such that x is a vertex of H and S is a subset of size s of the neighbors of x in some class C_i.

8: Upper bound the number of (x, S) by first picking S. Simplify the upper bound by using rough estimates.

Solution: Fixing S we have that any set $S \subset C_i$ has at most $(t-1)$ common neighbors, so the number of pairs (x, S) is at most

$$(t-1)\sum_{i=1}^{k}\binom{|C_i|}{s} \le (t-1)k\binom{\ell}{s} \le (t-1)k\frac{\ell^s}{s!}.$$

On the other hand, if we fix $x \in H$, then x has $d_i(x)$ neighbors in C_i, so the number of pairs (x, S) is

$$\sum_{(x,i)\in P}\binom{d_i(x)}{s}.$$

9: Use Jensen's inequality on the sum above and get a lower bound, simplify slightly.

Solution: The sum above has $|P|$ terms and the function $\binom{d_i(x)}{s}$ is convex, so by Jensen's inequality we have the number of pairs (x, S) is at least

$$|P|\binom{\frac{1}{|P|}\sum_{(x,i)\in P} d_i(x)}{s} = |P|\binom{2e(G)/|P|}{s} \ge |P|\frac{(2e(G)/|P| - (s-1))^s}{s!}.$$

10: Combine the two estimates for the number of pairs (x, S) and solving for $e(G)$ gives the theorem.

Solution:

$$|P|(2e(G)/|P| - (s-1))^s \le (t-1)k\ell^s$$
$$2e(G)/|P| - (s-1) \le (t-1)^{1/s}k^{1/s}\ell|P|^{-1/s}$$
$$2e(G) \le (t-1)^{1/s}k^{1/s}\ell|P|^{1-1/s} + (s-1)|P| \le (t-1)^{1/s}k^{1/s}\ell(2q\ell)^{1-1/s} + 2sq\ell$$

□

Notice in the following theorem that one of the forbidden graphs is linear in n.

Theorem 5. *Fix integers $k \geq 3$ and $s \geq 2$ and constants $c, \delta > 0$. Then, if G is a graph on n (large enough) vertices, that contains neither K_k nor $K_{s,t}$ where $t = \lceil cn \rceil$, then*

$$e(G) \leq c^{1/s}\left(1 - \frac{1}{k-1}\right)^{1-1/s}\frac{n^2}{2} + \delta n^2.$$

Proof. For $\epsilon = (\delta/8)^k$ and $m \geq \max\{\frac{8}{\delta}, k\}$, let $M = M(\epsilon, m)$ be as in the statement of the regularity lemma. We will prove the theorem for graphs G with $n \geq 2Ms/\delta$ many vertices. Let G be a graph on n vertices with no K_k nor $K_{s,t}$ subgraph and let $V_1, V_2, \ldots, V_r$ be the partition given by the regularity lemma with the parameters above.

11: Do a "cleaning up" G , denoted by G' and show that many edges remain. Use density between clusters $\delta/4$ for the clean-up. Show that $e(G) \leq e(G') + \frac{\delta}{2}n^2$.

Solution: We begin by "cleaning up" G as in the embedding lemma, i.e., remove all edges inside each cluster V_i and between any pair of distinct clusters that are not are $(\delta/8)^k$-regular or have density less than $\delta/4$. Let G' be the resulting graph. Therefore, the number of edges removed is at most

$$\left(\frac{1}{r} + 2\epsilon + \frac{\delta}{4}\right)n^2 \leq \left(\frac{\delta}{8} + \frac{\delta}{8} + \frac{\delta}{4}\right)n^2 \leq \frac{\delta}{2}n^2.$$

Thus

$$e(G) \leq e(G') + \frac{\delta}{2}n^2.$$

Now let R be the $(\epsilon, \delta/4)$-reduced graph, i.e., R is the graph with the clusters of G as vertices and two clusters are adjacent if they are $(\delta/8)^k$-regular and have density at least $\delta/4$ (in G).

12: Give an upper bound on the number of edges in R. (Hint: use K_k-free)

Solution: By the subgraph lemma we have that R is K_k-free and thus, by Turán's theorem, the number of edges in R is at most

$$\left(1 - \frac{1}{k-1}\right)\frac{r^2}{2}.$$

13: Apply (Lemma 4) with $q = e(R)$ and $\ell = \lceil n/r \rceil$ on G' to finish the proof.

Solution: Clearly G' must be $K_{s,t}$-free, so we can apply the previous lemma (Lemma 4) with $q = e(R)$ and $\ell = \lceil n/r \rceil$ to G' to get

$$e(G') \leq \frac{1}{2}\left(\left(1 - \frac{1}{k-1}\right)r^2\right)^{1-1/s}\ell^{2-1/s}(t-1)^{1/s}r^{1/s} + r^2\ell s.$$

Observe that $t - 1 \leq cn$ and $r^2\ell s \leq Mns \leq \frac{\delta}{2}n^2$, so we have

$$e(G') \leq c^{1/s}\left(1 - \frac{1}{k-1}\right)^{1-1/s}\frac{n^2}{2} + \frac{\delta}{2}n^2$$

□